Total No. of Printed Pages—7

2 SEM TDC MTMH (CBCS) C 3

2023

(May/June)

MATHEMATICS

(Core)

Paper : C–3

(Real Analysis)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* If $a \neq 0$, $b \neq 0$, then show that

$$\frac{1}{(ab)} = \left(\frac{1}{a}\right)\left(\frac{1}{b}\right), \; a, b \in \mathbb{R}$$ 3

(b) Prove that, if x is a rational number and y is an irrational number, then $x + y$ is an irrational number. 3

(c) If $f(x) = \dfrac{2x^2+3x+1}{2x-1}$ for $2 \le x \le 3$, find a constant M such that $|f(x)| \le M$ for all x satisfying $2 \le x \le 3$. 3

(d) State the supremum property of real numbers $\mathbb{R}$. 1

(e) If $S = \left\{1 - \dfrac{(-1)^n}{n};\ n \in \mathbb{N}\right\}$, then find inf S and sup S. 4

Or

Let S be a non-empty bounded set in $\mathbb{R}$. Let $a < 0$ and $aS = \{as : s \in S\}$. Prove that

$$\inf(aS) = a \sup S,\ \sup(aS) = a \inf S$$

(f) Prove that an upper bound u of a non-empty set S in $\mathbb{R}$ is the supremum of S if and only if for every $\varepsilon > 0$ there exists an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$. 3

(g) If x and y are any real numbers with $x < y$, then prove that there exists a rational number $r \in Q$ such that $x < r < y$. 3

(h) Prove that the set $\mathbb{R}$ of real numbers is not countable. 5

Or

If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed, bounded intervals such that the lengths $b_n - a_n$ of I_n satisfy $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then prove that the number ξ contained in I_n, $\forall\, n \in \mathbb{N}$ is unique.

(i) Show that if $a, b \in \mathbb{R}$ and $a \ne b$, then there exists ε-neighbourhoods U of a and V of b such that $U \cap V = \phi$. 5

Or

Prove that there does not exist a rational number r such that $r^2 = 2$.

2. (a) Define range of a real sequence. 1

(b) Write the limit point of the sequence (S_n), where

$$S_n = (-1)^n\left(1 + \frac{1}{n}\right),\ n \in \mathbb{N}$$ 2

(c) Every convergent sequence is bounded. Is the converse true? Justify. 1+2=3

(d) Prove that every bounded sequence has a limit point. 4

Or

Prove that $\underset{n\to\infty}{\text{Lt}}\left(\frac{1}{1+na}\right)=0,\ a>0.$

(e) Let the sequence $X=(x_n)$ converge to x. Prove that the sequence $(|x_n|)$ of absolute values converges to $|x|$. 4

(f) Define subsequence of a sequence of real numbers. 2

(g) If a sequence $X=(x_n)$ of real numbers converges to x, then prove that any subsequence $X'=(x_{n_k})$ of X also converges to x. 4

(h) Show that the sequence (e_n), where

$$e_n=\left(1+\frac{1}{n}\right)^n,\ n\in\mathbb{N}$$

is convergent. 5

Or

State and prove Bolzano-Weierstrass theorem.

(i) Show that the sequence (x_n), where

$$x_n=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!},\ n\in\mathbb{N}$$

is convergent. 5

Or

Show that the sequence (x_n), where

$$x_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

cannot converge.

3. (a) State the necessary condition for convergence of an infinite series. 1

(b) State True or False : 1

In convergent series, brackets may be inserted at will without affecting convergence but may not be removed.

(c) Discuss the convergence of a geometric series. 4

Or

Investigate the behaviour of the series whose nth term is $\sin\left(\frac{1}{n}\right)$.

(d) Prove that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $0 < p \leq 1$. 4

Or

Establish the convergence or divergence of the infinite series whose nth term is

$$\{(n^3+1)^{1/3} - n\}$$

(e) Define alternating series and conditionally convergent series. 2

(f) State the conditions of Leibnitz test. 2

(g) Test the convergence of the following (any *two*) : 3×2=6

(i) $\frac{1 \cdot 2}{3^2 \cdot 4^2} + \frac{3 \cdot 4}{5^2 \cdot 6^2} + \frac{5 \cdot 6}{7^2 \cdot 8^2} + \cdots$

(ii) $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \cdots$

(iii) $1 + \frac{4}{2!} + \frac{4^2}{3!} + \frac{4^3}{4!} + \cdots$

(iv) $\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} \cdots,$

x being a positive fraction.

★★★