## 1 SEM TDC MTMH (CBCS) C 2

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( Held in January/February, 2022 )

## MATHEMATICS

( Core )

Paper : C–2

### ( Algebra )

*Full Marks : 80*

*Pass Marks : 32*

*Time : 3 hours*

*The figures in the margin indicate full marks for the questions*

1. *(a)* Write the complex number $\sqrt{2}(1+i)$ in the polar form. 1

   *(b)* Find the equation whose roots are the $n$th power of the roots of the equation $x^2 - 2x\cos\theta + 1 = 0$. 2

   *(c)* Let $\operatorname{cis}\theta = \cos\theta + i\sin\theta$. If $x = \operatorname{cis}\alpha$, $y = \operatorname{cis}\beta$, $z = \operatorname{cis}\gamma$ and $x + y + z = xyz$, then show that

   $$1 + \cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = 0 \qquad 3$$

*( Turn Over )*

*Or*

If $\alpha$ denotes any $n$th roots of unity, then show that $1 + \alpha + \alpha^2 + \ldots\ldots + \alpha^{n-1} = 0$.

(d) Using De Moivre's theorem, find the expansions of $\cos n\theta$ and $\sin n\theta$ where $n \in \mathbb{N}$ and hence deduce the expansions of $\cos \alpha$ and $\sin \alpha$ in powers of $\alpha$.     4

2. (a) State whether true or false :     1

Union of two transitive relations is a transitive relation.

(b) Consider the functions $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = -2n$ and $g : \mathbb{N} \to \mathbb{R}$ defined by $g(n) = \dfrac{1}{n}$. Investigate the existence of $g \circ f$ justifying your assortion.     1

(c) Show that if $a \equiv b(\text{mod } m)$ and $c \equiv d(\text{mod } m)$, then

$$a + c \equiv b + d \ (\text{mod } m)$$     2

(d) Define an injective mapping. Show that the mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$ is injective.     2

(e) Let $n$ be a non-zero fixed integer. For any integers $a$ and $b$, define a relation $a \equiv b(\text{mod } n)$ if and only if $n$ divides $a - b$. Show that this relation is an equivalence relation.     4

( *Continued* )

*Or*

Show that intersection of two equivalence relations on a set is again an equivalence relation.

(f) State and prove the well ordering property of the set of positive integers.     4

*Or*

Show by the principle of mathematical induction that

$$1^2 + 2^2 + \ldots + n^2 = \frac{n}{6}(n+1)(2n+1)$$

(g) Let $f : A \to B$; $g : B \to C$; $h : C \to D$ be mappings. Show that

$$h \circ (g \circ f) = (h \circ g) \circ f$$     3

(h) Let g. c. d.$(a, b) = 1$. Show that

g. c. d. $(a + b, a^2 - ab + b^2) = 1$ or $3$     4

(i) Let $a$ and $b$ be two integers. Suppose either $a \neq 0$ or $b \neq 0$. Show that there exists a greatest common divisor $d$ of $a$, $b$ such that $d = ax + by$ for some integers $x$ and $y$ which is uniquely determined by $a$ and $b$.     4

( *Turn Over* )

( 4 )

**3.** (a) State whether true or false :    1

"Finding the parametric description of the solution set of a linear system is the same as solving the system."

(b) State which of the following statement/statements is/are false :    1

(i) The weights $c_1, c_2, ....., c_n$ in a linear combination $c_1 v_1 + c_2 v_2 + ..... c_n v_n$ of vectors $v_1, v_2, ...., v_n$ can not all be zero.

(ii) Another notation of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is $[a, b]$.

(iii) An example of a linear combination of vectors $v_1$ and $v_2$ is $\frac{1}{2} v_1$.

(iv) None of the above are true.

(c) Given $A = \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & 9 \end{bmatrix}$.

Find one non-trivial solution of $Ax = 0$.    2

(d) Show that the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are linearly dependent.    2

( 5 )

(e) Give the geometrical interpretation of span$\{u, v\}$ where

$$u = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$$

Indicate the subspace represented by the span.    2

(f) Define linear independence of vectors. Show that the columns of the matrix

$$A = \begin{bmatrix} 4 & 3 \\ -2 & -3 \\ 6 & 9 \end{bmatrix} \text{ are linearly independent.}$$

   1+2=3

(g) Show that if an indexed set $S = \{v_1, ....., v_n\}$ with $n \geq 2$, is linearly dependent and $v_1 \neq 0$, then some $v_j$ with $j > 1$ is a linear combination of the preceding vectors $v_1, ....., v_{j-1}$.    4

(h) Transform the augmented matrix represented by the linear system,

$$x_1 + 3x_2 + x_3 = 0$$
$$-4x_1 - 9x_2 + 2x_3 = 0$$
$$-3x_2 - 6x_3 = 0$$

(i) to Echelon form indicating the forward phase of row operations.

*Or*

Find the characteristic equation of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

and the eigenvalues.

(g) Given a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = Ax$ where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Find $T(u)$, $T(v)$ and $T(u + v)$ where $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and interpret the effect of the transformation geometrically. 2+2=4

(h) Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

4

*Or*

If $v_1, \ldots\ldots, v_p$ are eigenvectors

*(i)* Let $A$ be an invertible matrix. Show that

  *(i)* $(A^{-1})^{-1} = A$

  *(ii)* $(AB)^{-1} = B^{-1}A^{-1}$       2+3=5

*Or*

Let $v_1, ...., v_p \in \mathbb{R}^n$. Show that the set of all linear combinations of $v_1, ...., v_p$ is a subspace of $\mathbb{R}^n$.

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