**3 SEM TDC MTMH (CBCS) C 6**

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( Held in January/February, 2022 )

MATHEMATICS

( Core )

Paper : C–6

**( Group Theory—I )**

*Full Marks* : 80

*Pass Marks* : 32

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

**1.** (a) What is the inverse of the element 13 in $Z_{20}$? 1

(b) List the elements of $U(20)$. 1

(c) Let G be a group and $a, b \in G$ such that $a^3 = e$, $aba^{-1} = b^2$. Find $O(b)$. 2

(d) Let G be a group, then prove that

$$(ab)^{-1} = b^{-1}a^{-1}, \forall\, a, b \in G$$ 2

(e) In $D_4$, find all elements $X$ such that

(i) $X^3 = V$

(ii) $X^3 = R_{90}$

(iii) $X^3 = R_0$

(iv) $X^2 = R_0$ 4

Or

Construct a complete Cayley table for $D_3$.

(f) Prove that the set $G = \{1, 2, 3, 4, 5, 6\}$ is a finite abelian group of order 6 with respect to multiplication modulo 7. 5

2. (a) Let $H$ and $K$ be two subgroups of a group G. Then, write the condition such that $H \cup K$ may be a subgroup of G. 1

(b) Define index of a subgroup in a group. 2

(c) Prove that a non-empty subset $H$ of a finite group $G$ is a subgroup of $G$ iff $HH = H$. 4

(d) Define normalizer of an element in a group G and also show that $N(a)$ is a subgroup of the group G where $a \in G$. 4





Or

Prove that $O(C(a)) = 1$ if and only if $a \in Z(G)$.

(e) Prove that the relation of conjugacy is an equivalence relation. 4

3. (a) Write all the subgroups of a cyclic group of order 12. 1

(b) State Fermat's little theorem. 1

(c) Prove that a group of prime order has no proper subgroup. 2

(d) Give an example of a cyclic group whose order is not prime. 2

(e) Let $G$ be a group and $H$ be a subgroup of G. Let $a, b \in G$. Then show that

(i) $Ha = Hb$ iff $ab^{-1} \in H$

(ii) $Ha$ is a subgroup of $G$ iff $a \in H$ 4

(f) Let $H$ be a subgroup of a finite group G. Then prove that the order of $H$ divides the order of G. 5

(g) Prove that an infinite cyclic group has exactly two generators. 5

*Or*

Prove that the order of a finite cyclic group is equal to the order of its generator.

4. (a) State Cauchy's theorem for finite abelian group. 1

(b) Prove that quotient group of an abelian group is abelian. 2

(c) Prove that every subgroup of a cyclic group is normal. 3

(d) Let $H$ and $K$ be two subgroups of a group $G$. Then prove that $HK$ is a subgroup of $G$ if $K$ is normal subgroup of $G$. Also if $H$ and $K$ both are normal subgroups, then $HK$ is also normal subgroup of $G$. 4

(e) If $G_1$ and $G_2$ are groups, then prove that

(i) identity is the only element common to $G_1 \times \{e_2\}$ and $\{e_1\} \times G_2$





(ii) every element of $G_1 \times G_2$ can be uniquely expressed as the product of an element in $G_1 \times \{e_2\}$ by an element in $\{e_1\} \times G_2$

(iii) $G_1 \times G_2 \cong G_2 \times G_1$ 1+2+2=5

*Or*

Let $H$ be a subgroup of a group $G$ such that $x^2 \in H, \forall x \in G$. Then prove that $H$ is normal subgroup of $G$. Also prove that $G/H$ is abelian. 5

5. (a) Let $H$ be a normal subgroup of a group $G$ and $f: G \to G/H$ such that $f(x) = Hx$, $\forall x \in G$. Then prove that $f$ is an epimorphism. 2

(b) Let $f$ be a homomorphism from a group $G$ into a group $G'$. Then prove that

(i) $f(a^{-1}) = [f(a)]^{-1}, \forall a \in G$

(ii) if $O(a)$ is finite, then $O(f(a)) \mid O(a)$ where $a \in G$ 3

(c) Let $H$ and $K$ be two normal subgroups of a group $G$ such that $H \subseteq K$. Then prove that $\frac{G}{K} \cong \frac{G/H}{K/H}$. 5

(d) Prove that the necessary and sufficient condition for a homomorphism of a group $G$ onto a group $G'$ with kernel $K$ to be an isomorphism is that $K = \{e\}$. 5

*Or*

State and prove Cayley's theorem.

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